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THE RIEMANN FUNCTION OF THE CAUCHY PROBLEM FOR A SECOND-ORDER HYPERBOLIC EQUATION WITH A PERIODIC COEFFICIENT

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Abstract. The Cauchy problem for the second-order hyperbolic equation $\frac{\partial^2 U}{\partial t^2} - p(t)U = \frac{\partial^2 U}{\partial x^2} - p(x)U$ with the initial conditions $U|_{t=t_0} = 0$, $\frac{\partial U}{\partial t}\big|_{t=t_0} = f(x)$ is considered. An explicit form of the Riemann function of this problem is found.

Keywords: The Cauchy problem, a second-order hyperbolic equation, the Riemann function, Hill equation, method of successive approximations.

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1 Introduction and statement of the problem

Let $p(x)$ be a real continuously differentiable function on the line $(-\infty, +\infty)$, which $p(x+1) = p(x)$, and let $f(x)$ be a real twise- differentiable function with bounded support. Under these assumptions on the functions $p(x)$ and $f(x)$, we will consider the Cauchy problem:

$$\frac{\partial^2 U}{\partial t^2} - p(t)\,U = \frac{\partial^2 U}{\partial x^2} - p(x)\,U, \tag{1}$$

$$U|_{t=t_0} = 0,\ \frac{\partial U}{\partial t}\bigg|_{t=t_0} = f(x), \tag{2}$$

where t_0 is an arbitrary constant. It is known that one of the main tools for studying the Cauchy problem for a second-order hyperbolic equation is the application of the Riemann function method (Riemann, 1948). To apply the method, it is necessary to construct the Riemann function $R(x,t,X,T)$, which is a twice continuously differentiable solution of the equation

$$\frac{\partial^2 R}{\partial t^2} - p(t)\,R = \frac{\partial^2 R}{\partial x^2} - p(x)\,R \tag{3}$$

which takes the value on the characteristics $x - X = \pm(t-T)$. It is known that a solution to equation (3) with the listed properties exists and is unique. Note that a general method for constructing the Riemann does not exist. In this direction, we note the paper, in which an extensive analysis of six certain methods for creating Riemann functions of particular types of equations is given.

It is well known that , using the Riemann function, the solution to problem (1), (2) can be represented by the formula

$$U(X,T)=\pm\frac{1}{2}\int\limits_{X-T+t_0}^{X+T-t_0} f(x)\,R(x,t_0,X,T)\,dx, \tag{4}$$

where the sign corresponds to the case of $\pm(T-t_0)>0$.

In this paper, using the method of separation of variables, we find a different form of a solution to problem (1), (2). If we could solve this problem by some other method, a comparison of the two solutions would give $R(x,t_0,X,T)$, when x lies between $\pm(T-t_0)>0$; as t_0 is arbitrary, this would give $R(x,t,X,T)$, whenever $X-x$ lies between $\pm(T-t)$. The obtained results can be used to construct transformation operators for the perturbed Hill equations (see Firsova (1975)).

2 Method of construction of the Riemann function

We consider the Hill's equation

$$-y''+p(x)\,y=\lambda y, -\infty<x<+\infty, \tag{5}$$

where the real function $p(x)$ satisfies the conditions

$$p(x)\in C^{(1)}(-\infty,+\infty)\,,p(x+1)=p(x)\,. \tag{6}$$

We introduce the solutions $\varphi(x,\lambda)$ and $\theta(x,\lambda)$ of equation (5), that satisfy the conditions $\varphi(0,\lambda)=0,\varphi'(0,\lambda)=1$ and $\theta(0,\lambda)=1,\theta'(0,\lambda)=0$. We put $F(\lambda)=\frac{\varphi'(1,\lambda)+\theta(1,\lambda)}{2}$. Let $e^{\pm ik}=F(\lambda)\pm i\sqrt{1-F^2(\lambda)}$ and functions $\Psi_{1,2}(x,k)=e^{\pm ikx}\chi(x,k)\,,$ where $\chi(x+1,k)=\chi(x,k)$, are the normalized Floquet solutions(see) of the equation (5). In Korovina et al. (2021), the following form of the expansion theorem for a function $f(x)\in L_2(-\infty,+\infty)$ was obtained

$$f(x)=\frac{1}{2\pi}\int\limits_{-\infty}^{+\infty}\Psi_2(x,k)\int\limits_{-\infty}^{+\infty}f(y)\,\Psi_1(y,k)\,dydk \tag{7}$$

Let us now return to problem (1) - (2). Applying the d'Alembert's formula (see also Korovina et al. (2021)), we obtain the following integral equation, which is equivalent to problem (1), (2):

$$U(x,t)=\frac{1}{2}\int\limits_{x-(t-t_0)}^{x+(t-t_0)} f(\xi)\,d\xi+\frac{1}{2}\int\limits_{t_0}^{t}d\tau\int\limits_{x-(t-\tau)}^{x+(t-\tau)}[p(\tau)-p(\xi)]\,U(\xi,\tau)\,d\xi. \tag{8}$$

The integral equation (8) is an equation of Volterra type and can therefore be solved by the method of successive approximations.

On the other hand, it is easy to check that for any $A>0$ a function of the form

$$U_A(x,t)=\int\limits_{-A}^{A}\{f_1(k)\,\Psi_1(t,k)+f_2(k)\,\Psi_2(t,k)\}\,\Psi_1(x,k)\,dk$$

is a solution to equation (5), where $f_1(k)$ and $f_2(k)$ are functions to be determined. Based on the formula (7), we put $\tilde{f}(k) = \frac{1}{2\pi} \int\limits_{-\infty}^{+\infty} f(x)\Psi_2(x,k)\,dx$, $f_A(x) = \int\limits_{-A}^{A} \tilde{f}(k)\Psi_1(x,k)\,dk$ and require that the solution satisfies the initial conditions

$$U_A\big|_{t=t_0} = 0, \left.\frac{\partial U_A}{\partial t}\right|_{t=t_0} = f_A(x) \tag{9}$$

The latter conditions are deliberately fulfilled, if

$$\begin{cases} f_1(k)\Psi_1(t_0,k) + f_2(k)\Psi_2(t_0,k) = 0, \\ f_1(k)\Psi_1'(t_0,k) + f_2(k)\Psi_2'(t_0,k) = \tilde{f}(k). \end{cases}$$

Solving the last system of equations for $f_1(k)$ and $f_2(k)$, we find that

$$f_1(k) = -\frac{\tilde{f}(k)\Psi_2(t_0,k)}{W(k)}, f_2(k) = \frac{\tilde{f}(k)\Psi_1(t_0,k)}{W(k)},$$

where $W(k) = \frac{i\sin k}{F'(\lambda)}$ denotes the Wronskian of solutions $\Psi_1(x,k)$, $\Psi_2(x,k)$, (see Firsova (1975)). Therefore, the representation

$$U_A(X,T) = \int\limits_{-A}^{A} \frac{\tilde{f}(k)}{W(k)} \{\Psi_1(t_0,k)\Psi_2(T,k) - \Psi_1(T,k)\Psi_2(t_0,k)\}\Psi_1(X,k)\,dk$$

is valid. Using the equality $\tilde{f}(k) = \frac{1}{2\pi}\int\limits_{-\infty}^{+\infty} f(x)\Psi_2(x,k)\,dx$, we finally obtain

$$U_A(X,T) = \frac{1}{2\pi}\int\limits_{-\infty}^{+\infty} f(x) \int\limits_{-A}^{A} W^{-1}(k)\{\Psi_1(t_0,k)\Psi_2(T,k) -$$

$$-\Psi_1(T,k)\Psi_2(t_0,k)\}\Psi_1(X,k)\Psi_2(x,k)\,dkdx. \tag{10}$$

Thus, function (10) is a solution of the problem (1), (9).

Further, proceeding with the last problem in exactly the same way as it was done with problem (1), (2), instead of equation (8) we obtain

$$U_A(X,T) = \frac{1}{2}\int\limits_{X-(T-t_0)}^{X+(T-t_0)} f_A(\xi)\,d\xi + \frac{1}{2}\int\limits_{t_0}^{T} d\tau \int\limits_{X-(T-\tau)}^{X+(T-\tau)} [p(\tau) - p(\xi)]\,U_A(\xi,\tau)\,d\xi. \tag{11}$$

Since $f_A(x)$ converges in the norm of the space $L_2(-\infty,+\infty)$ to the function $f(x)$ for $A \to +\infty$, then using the method of successive approximations from (8), (11), we find that $U_A(X,T)$ is uniformly convergent to the function $U(X,T)$ in each finite region of variation of the variables X and T.

Now we consider the function $\Gamma(x,t,y,z)(X,T)$, defined by the formula

$$\Gamma(x,t,y,z) = \frac{1}{2\pi}\int\limits_{-\infty}^{+\infty} W^{-1}(k)\,[\Psi_1(x,k)\Psi_2(t,k) -$$

$$-\Psi_1(t,k)\Psi_2(x,k)]\,\Psi_1(y,k)\Psi_2(z,k)\,dk, \tag{12}$$

where $W(k)$ is the Wronskian of solutions $\Psi_1(x,k)$, $\Psi_2(x,k)$. As shown in Firsova (1975), for all x,t,y,z the function $\Gamma(x,t,y,z)$ and its first-order partial derivatives are uniformly bounded.

Moreover, this function vanishes for $\pm(z-y) > t-x$. Letting $A \to +\infty$ in the formulas (10), (11), we conclude that the solution of the problem (1), (2) admits the representation

$$u(X,T) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} f(x) \int_{-\infty}^{+\infty} W^{-1}(k)\{\Psi_1(t_0,k)\Psi_2(T,k) -$$

$$-\Psi_1(T,k)\Psi_2(t_0,k)\}\Psi_1(X,k)\Psi_2(x,k)\,dkdx$$

Comparing this representation with (4), we obtain the following theorem.

Theorem 1. *The Riemann function of the problem (1) - (2) admits the representation*

$$R(x,t,X,T) = \pm\frac{1}{\pi}\int_{-\infty}^{+\infty} W^{-1}(k)\{\Psi_1(t,k)\Psi_2(T,k) -$$

$$-\Psi_1(T,k)\Psi_2(t,k)\}\Psi_1(X,k)\Psi_2(x,k)\,dk,$$

where the $\pm$sing sign corresponds to the case $\pm(T-t) > 0$.

Remark 1. *It follows from the proof of the theorem that $R(x,t,X,T) = \pm 2\Gamma(t,T,X,x)$. As it is clear from the previous one, under one of the conditions $X-T+t < x < X+T-t$ and $X+T-t < x < X-T+t$, the function $R(x,t,X,T)$ and its first-order partial derivatives are uniformly bounded:*

$$|R(x,t,X,T)| + \left|\frac{\partial R(x,t,X,T)}{\partial x}\right| + \left|\frac{\partial R(x,t,X,T)}{\partial t}\right| +$$
$$\left|\frac{\partial R(x,t,X,T)}{\partial X}\right| + \left|\frac{\partial R(x,t,X,T)}{\partial T}\right| \le C, C = const.$$

In addition, it follows from the properties of the Riemann function that under the condition $x - x = \pm(T-t)$, the equality $\Gamma(t,T,X,x) = \frac{1}{2}$ holds.

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